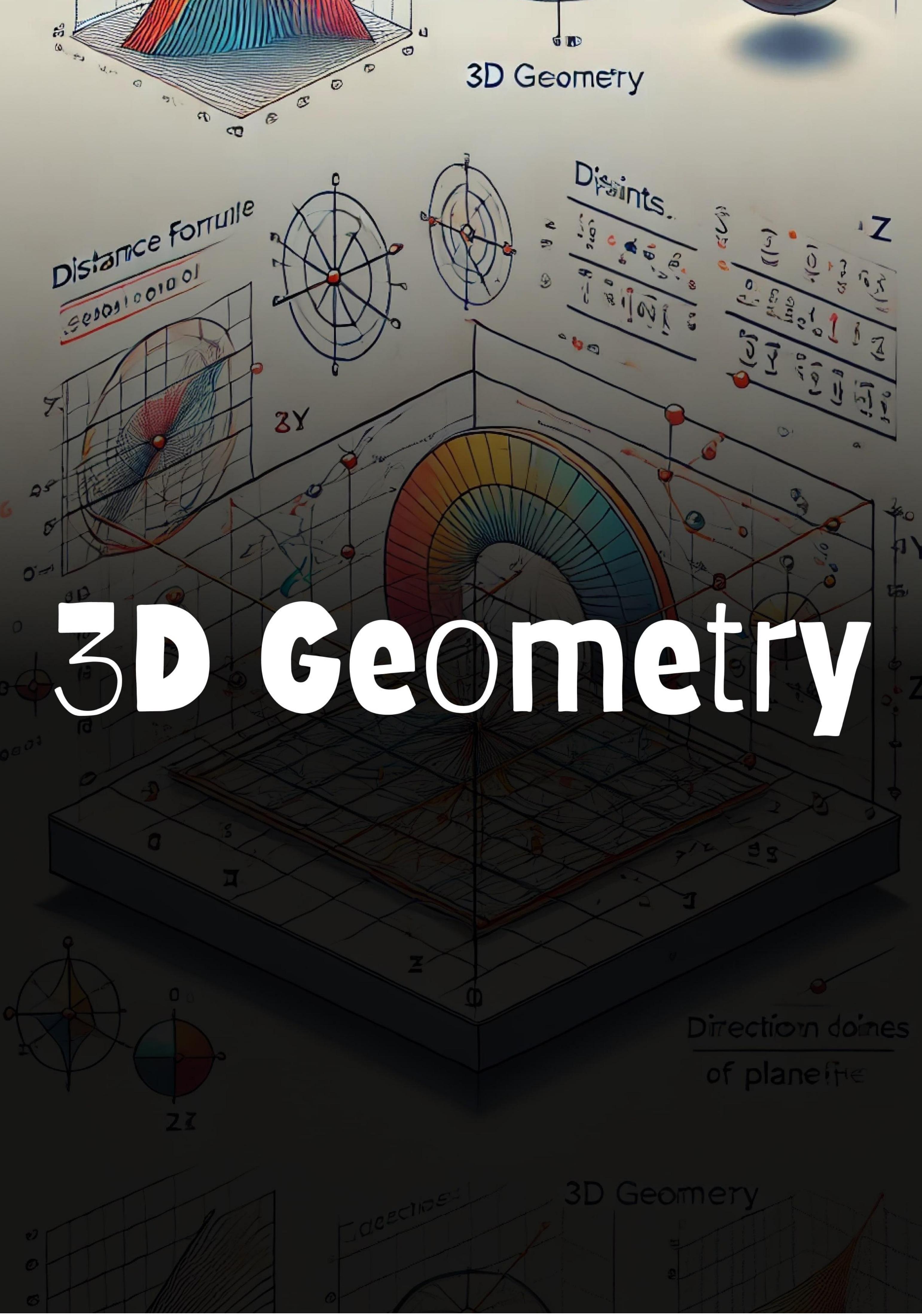
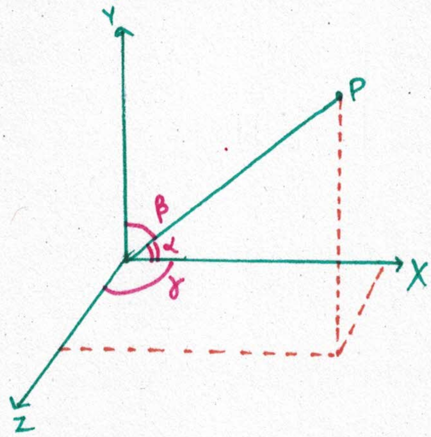


3D Geometry

3D-GEOMETRY



Let $\vec{OP} = a\hat{i} + b\hat{j} + c\hat{k}$

then the cosines of angles made by this vector with +ve x-axis and y-axis and z-axis are known as direction cosines of $\vec{OP}$.

Let α, β, γ are direction angles with x-axis y-axis and z axis respectively, then $\cos\alpha$, $\cos\beta$ and $\cos\gamma$ are known as direction cosines.

$\vec{OP} \cdot \hat{i} = |\vec{OP}||\hat{i}| \cos\alpha$

$a = \sqrt{a^2+b^2+c^2} \cos\alpha$

$\cos\alpha = a / \sqrt{a^2+b^2+c^2}$

$\cos\beta = m = b / \sqrt{a^2+b^2+c^2}$

$\cos\gamma = n = c / \sqrt{a^2+b^2+c^2}$

POINTS TO REMEMBER

1. $l^2 + m^2 + n^2 = 1$
   i.e. $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$
2. $\sin^2\alpha + \sin^2\beta + \sin^2\gamma = 2$
3. $\cos 2\alpha + \cos 2\beta + \cos 2\gamma = -1$
4. Direction cosines are simply components of unit vector
5. The DC's of $\vec{PO}$ is $(\pi-\alpha), (\pi-\beta)$ & $(\pi-\gamma)$

DIRECTION RATIO

Let a, b, c are 3 no's proportional to DC then a, b, c are called DRs of the given vector

$\vec{OP} = a\hat{i} + b\hat{j} + c\hat{k}$, then a, b, c are DR

whereas, $\dfrac{a}{\sqrt{a^2+b^2+c^2}}, \dfrac{b}{\sqrt{a^2+b^2+c^2}}, \dfrac{c}{\sqrt{a^2+b^2+c^2}}$ are DC i.e. $\frac{a}{l} = \frac{b}{m} = \frac{c}{n} = \lambda$

NOTE

1. If a line having DC l, m, n then, that line is travelling along a vector - $l\hat{i} + m\hat{j} + n\hat{k}$

2. Direction ratio of a line joining $\langle x_1, y_1, z_1 \rangle$ & $\langle x_2, y_2, z_2 \rangle$ will be $\langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$

3. Angle between the 2 lines l_1, l_2 and having DC l_1, m_1, n_1 & l_2, m_2, n_2 will be given as -

$$\cos\theta = l_1 l_2 + m_1 m_2 + n_1 n_2$$

$$\cos\theta = \frac{a_1 a_2 + b_1 b_2 + c_1 c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

where $\langle a_1, b_1, c_1 \rangle$ & $\langle a_2, b_2, c_2 \rangle$ are DR of l_1 and l_2 respectively

4. If two lines l_1 and l_2 are mutually perpendicular, then -

$$l_1 l_2 + m_1 m_2 + n_1 n_2 = 0$$

$$\Rightarrow a_1 a_2 + b_1 b_2 + c_1 c_2 = 0$$

5. If two lines are parallel, then

$$\frac{l_1}{l_2} = \frac{m_1}{m_2} = \frac{n_1}{n_2} \Rightarrow \frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$$

Ques: Find the DC of a line perpendicular to two lines having DR:- $\langle 1, 2, 3 \rangle$ and $\langle -2, 1, 4 \rangle$

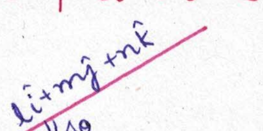
Sol:
$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 3 \\ -2 & 1 & 4 \end{vmatrix} = \hat{i}(8-3) - \hat{j}(4+6) + \hat{k}(1+4)$$
$$= 5\hat{i} - 10\hat{j} + 5\hat{k}$$

$$DC's = \frac{5}{\sqrt{150}}, \frac{-10}{\sqrt{150}}, \frac{5}{\sqrt{150}}$$

$$= \frac{1}{\sqrt{6}}, \frac{-2}{\sqrt{6}}, \frac{1}{\sqrt{6}}$$

Ques: Let a variable line having DCs l, m, n and $l + \Delta l$, $m + \Delta m$, $n + \Delta n$ in 2 adjacent position if the angle between these 2 adjacent position is $\Delta\theta$, then prove that $\sum(\Delta l)^2 = (\Delta\theta)^2$

Sol:



$$|\Delta l\hat{i} + \Delta m\hat{j} + \Delta n\hat{k}| = \sqrt{1 + 1 + 2(1)(1)\cos\Delta\theta}$$

$$(\Delta l)^2 + (\Delta m)^2 + (\Delta n)^2 = (\Delta\theta)^2$$

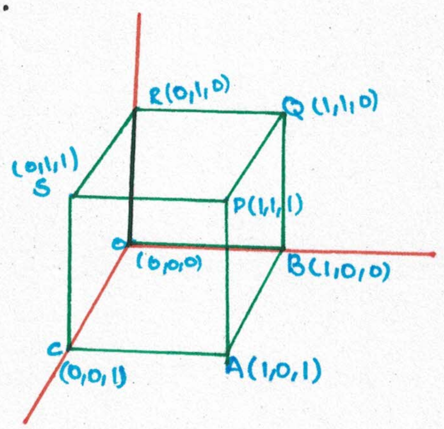
$\Rightarrow \sqrt{\Delta l^2 + \Delta m^2 + \Delta n^2} = \sqrt{2(1-\cos\Delta\theta)}$

$\Rightarrow \Delta l^2 + \Delta m^2 + \Delta n^2 = 2\sin^2\frac{\theta}{2}$

$\Rightarrow \Delta l^2 + \Delta m^2 + \Delta n^2 = \Delta\theta^2$

Ques: If a line with DC l, m & n makes an angle α, β, γ & δ with four diagonals of a cube. Find $\cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta$

Sol:



$\vec{OP} = \hat{i} + \hat{j} + \hat{k}$ $\quad\quad$ $\vec{AR} = \hat{i} - \hat{j} + \hat{k}$

$\cos\alpha = \frac{l+m+n}{\sqrt{3}}$ $\quad\quad$ $\cos\beta = \frac{l+m+n}{\sqrt{3}}$

$\cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta$

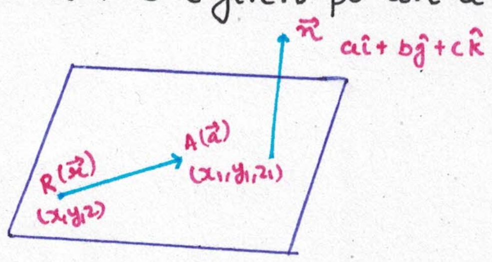
$= \frac{4}{3}(l^2+m^2+n^2)$

$= \frac{4}{3}$

PLANE

VECTOR FORM OF A PLANE

Let $A(\vec{a})$ is a given pt. in a plane whose normal vector is $\vec{n}$, Then,



$\vec{AR} \perp \vec{n}$

$\vec{AR} \cdot \vec{n} = 0$

$(\vec{r} - \vec{a}) \cdot \vec{n} = 0$

$\vec{r} \cdot \vec{n} - \vec{a} \cdot \vec{n} = 0$

$\vec{r} \cdot \vec{n} = \vec{a} \cdot \vec{n}$

$\boxed{\vec{r} \cdot \vec{n} = d}$

$ax + by + cz = ax_1 + by_1 + cz_1$

$a(x-x_1) + b(y-y_1) + c(z-z_1) = 0$

Cartesian Equation of plane

$ax + by + cz + W = 0$

$W = -(ax_1 + by_1 + cz_1)$

POINTS TO REMEMBER

Coefficient of x, y, z in cartesian form of a plane is DR of the normal vector of the plane

If 2 planes $P_1 : a_1x + b_1y + c_1z + d_1 = 0$ $P_2 : a_2x + b_2y + c_2z + d_2 = 0$ are parallel then, $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2} \neq \frac{d_1}{d_2}$

If $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2} = \frac{d_1}{d_2}$, then both P_1 & P_2 are overlapped planes

If P_1 & P_2 are mutually perpendicular planes then,
$a_1a_2 + b_1b_2 + c_1c_2 = 0$

Ques: Find the equation of plane passing through $\langle 1, 0, -2 \rangle$ are perpendicular to the planes $P_1 - 2x + y - z = 2$ and $P_2 - x - y - z = 3$.

Sol: $\vec{n} = x_1, y_1, z_1$

$\vec{n_1} \cdot \vec{n} = 2x_1 + y_1 - z_1 = 0$

$\vec{n_2} \cdot \vec{n} = x_1 - y_1 - z_1 = 0$

$3x_1 - 2z_1 = 0$

$z_1 = \frac{3x_1}{2}$

$2x_1 + y_1 - \frac{3x_1}{2} = 0$

$4x_1 - 3x_1 = 0$

$$\vec{P_1} \times \vec{P_2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & -2 \\ 1 & -1 & -1 \end{vmatrix} = \hat{i}(-1-2) - \hat{j}(-2+2) + \hat{k}(-2-1)$$

$= -3\hat{i} - 3\hat{k}$

$(x\hat{i} + y\hat{j} + z\hat{k}) \cdot (-3\hat{i} - 3\hat{k}) = (1, 0, -2)(-3, 0, -3)$

Ques: Find the equation of a plane which contains $\langle 3, 1, 1 \rangle$ and which is parallel to the plane $2x + 3y + z + 9 = 0$

Sol: $\vec{n} = 2\hat{i} + 3\hat{j} + \hat{k}$

$2(x-3) + 3(y-1) + 1(z-1) = 0$

$2x + 3y + z - 10 = 0$

Ques: The foot of the normal from origin on a plane is $\langle 1, 2, 3 \rangle$. Then find the equation of plane.

Sol: DR's $\longrightarrow$ 1, 2, 3

$1(x-1) + 2(y-2) + 3(z-3) = 0$

$x + 2y + 3z = 14$

Ques: Two planes $x + 2y - 3z = 0$ and $2x + y + z - 3 = 0$ are given then find the equation of a plane perpendicular to P_1 & P_2 and passing through $\langle 2, 2, 1 \rangle$. Also find the DR's of their line of intersection.

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & -3 \\ 2 & 1 & 1 \end{vmatrix} = \hat{i}(2+3) - \hat{j}(1+6) + \hat{k}(1-4)$$

$$= 5\hat{i} - 7\hat{j} - 3\hat{k}$$

$$5(x-2) - 7(y-2) - 3(z-1) = 0$$

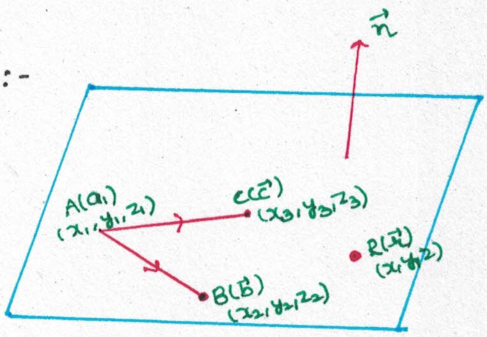
$$\Rightarrow \quad 5x - 7y - 3z + 7 = 0$$

EQUATION OF PLANE PASSING THROUGH 3 POINTS

If we wish to find the equation of a plane containing the pts. $A(\vec{a})$, $B(\vec{b})$, $C(\vec{c})$ then, first we have to find a normal vector perpendicular to the plane and can be obtained by $\vec{AB} \times \vec{AC}$

Now assume a variable vector $R(\vec{r})$

The equation of plane will be given as:-



$$\vec{AR} \cdot \vec{n} = 0$$

$$\vec{AR} \cdot [\vec{AB} \times \vec{AC}] = 0$$

$$[\vec{AR} \quad \vec{AB} \quad \vec{AC}] = 0 \quad \} \text{ Vetor Form}$$

$$\left.\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix} = 0 \quad \right\} \text{ Cartesian Form}$$

Ques: Find the equation of a plane containing $A\langle 1, 1, 1\rangle$, $B\langle 3, 5, 7\rangle$ and $C\langle 0, 2, 0\rangle$

Sol: $\vec{AB} = \langle 2, 4, 6\rangle \qquad \vec{AC} = \langle -1, 1, -1\rangle$

$$\vec{n} \begin{vmatrix} x-1 & y-1 & z-1 \\ 2 & 4 & 6 \\ -1 & 1 & -1 \end{vmatrix} = 0$$

$$\Rightarrow \quad (x-1)(-4-6) - (y-1)(-2+6) + (z-1)(2+4) = 0$$

$$\Rightarrow \quad -10x + 10 + 4y + 4 + 6z - 6 = 0$$

$$\Rightarrow \quad 5x + 2y - 3z - 4 = 0$$

Ques: Consider the 3 lines $\vec{r}_1 = 3\hat{i} - \hat{j} + 2\hat{k} + \lambda(2\hat{i} + 4\hat{j} - \hat{k})$, $\vec{r}_2 = \hat{i} + \hat{j} - 3\hat{k} + \mu(4\hat{i} + 2\hat{j} + 4\hat{k})$, $\vec{r}_3 = 3\hat{i} + 2\hat{j} - 2\hat{k} + t(2\hat{i} + \hat{j} + 2\hat{k})$, which one of the following pairs are in the same plane?

(A) $L_1 L_3$ (B) $L_2 L_4$ (C) $L_2 L_3$ (D) All are in same plane

$3+2\lambda = 1+4\mu$ $\qquad$ $-1+4\lambda = 1+2\mu$ $\qquad$ $2-1$ $\qquad$ $-3+4$

$2\lambda - 4\mu + 2 = 0$ $\qquad$ $4\lambda - 2\mu = 2$ $\qquad$ 1 $\qquad$ 1

$2\lambda - 8\lambda + 4 + 2 = 0$ $\qquad$ $2\lambda - \mu = 1$ $\qquad$ Intersecting

$-6\lambda + 6 = 0$ $\qquad$ $\mu = 2\lambda - 1$

$\lambda = 1$ $\qquad$ $\mu = 1$

$$\begin{vmatrix} 2 & -2 & 5 \\ 2 & 4 & -1 \\ 4 & 2 & 4 \end{vmatrix} = 0 \Rightarrow L_1 \,\&\, L_2 \text{ are coplanar}$$

$L_1 \,\&\, L_3$

$$\begin{vmatrix} 0 & -3 & 4 \\ 2 & 4 & -1 \\ 2 & 1 & 2 \end{vmatrix}$$

$3(4+2) + 4(2-8) \neq 0 \Rightarrow L_1 \,\&\, L_2$ are not coplanar

Ans – (B), (C)

Ques: Find the equation of a plane which contain the ΔABC whose P.V are A $\langle 2, 1, 3\rangle$, B $\langle 1, 0, -1\rangle$, C $\langle 0, 0, 5\rangle$

Sol: $\vec{AB} = \langle -1, -1, -4\rangle$ $\qquad$ $\vec{AC} = \langle -2, -1, 2\rangle$

$$\begin{vmatrix} x-2 & y-1 & z-3 \\ -1 & -1 & -4 \\ -2 & -1 & 2 \end{vmatrix} = 0$$

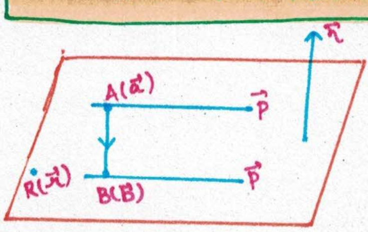
$\Rightarrow (x-2)(-2-4) - (y-1)(-2-8) + (z-3)(1-2) = 0$

$\Rightarrow (x-2)(-6) - (y-1)(-10) + (z-3)(-1) = 0$

$\Rightarrow -6x + 12 + 10y - 10 - z + 3 = 0$

$\Rightarrow 6x - 10y + z = 5$

EQUATION OF A PLANE CONTAINING TWO PARALLEL LINES



$\vec{n} = \vec{AB} \times \vec{p}$

$\vec{AR} \cdot \vec{n} = 0$

$$\boxed{\vec{AR} \cdot [\vec{AB} \times \vec{p}] = 0}$$

$$\boxed{[\vec{AR} \;\; \vec{AB} \;\; \vec{p}] = 0}$$

Ques: Find the equation of a plane passing through $\langle 3, 4, 5\rangle$ and contains the line $\vec{r} = \hat{i} + \hat{j} + \hat{k} + \lambda(3\hat{i} - \hat{j} + 2\hat{k})$

Sol: $\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 4 & 5 \\ 1 & 1 & 1 \end{vmatrix}$ $\qquad$ $\vec{n} = \hat{i}(-1) - \hat{j}(-2) + \hat{k}(-1) = 0$

$= \hat{i} - 2\hat{j} + \hat{k} = 0$

$1(x-3) - 2(y-4) + 1(z-5) = 0$

$x - 2y + z - 3 + 8 - 5 = 0$

$x - 2y + z = 0$

$$\begin{vmatrix} (x-3) & (y-4) & (z-5) \\ 2 & 3 & 4 \\ 3 & -1 & 2 \end{vmatrix}$$

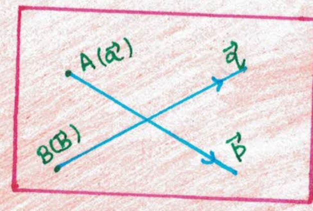
$\Rightarrow (x-3)(10) - (y-4)(4-12) + (z-5)(-2-9)$

$\Rightarrow 10x + 8y - 11z - 30 - 32 + 55 = 0$

$\Rightarrow 10x + 8y - 11z + 3 = 0$

POINTS TO REMEMBER

Two intersecting line always determine a unique plane. Let $L_1: \vec{a} + \lambda\vec{p}$ and $L_2: \vec{b} + \lambda\vec{q}$ are intersecting lines.



$[\vec{AB} \quad \vec{p} \quad \vec{q}] = 0$

$\Rightarrow [\vec{b} - \vec{a} \quad \vec{p} \quad \vec{q}] = 0$

$\Rightarrow [\vec{b} \quad \vec{p} \quad \vec{q}] - [\vec{a} \quad \vec{p} \quad \vec{q}] = 0$

$\Rightarrow [\vec{a} \quad \vec{p} \quad \vec{q}] = [\vec{b} \quad \vec{p} \quad \vec{q}]$

Equation of plane containing both lines,

$[\vec{AR} \quad \vec{p} \quad \vec{q}] = 0$

$\Rightarrow \vec{AR}, \vec{p}, \vec{q}$ are coplanar

$\Rightarrow \vec{AR} = \lambda\vec{p} + \mu\vec{q}$

$$\boxed{\vec{r} = \vec{a} + \lambda\vec{p} + \mu\vec{q}}$$

The above equation represents the parametric equation of a plane containing pt. A and parallel to the two planes containing two non-collinear vectors $\vec{p}$ and $\vec{q}$

Ques: $\vec{r} = \hat{i} - 2\hat{j} + \lambda(2\hat{i} - \hat{j} + 3\hat{k}) + \mu(3\hat{i} + 4\hat{j} - \hat{k})$ is an equation of a plane in parametric form then convert it into vector & cartesian form.

Sol: $\begin{vmatrix} x-1 & y+2 & z \\ 2 & -1 & 3 \\ 3 & 4 & -1 \end{vmatrix} = 0$

$(x-1)(1-12) - (y+2)(-2-9) + z(8+3) = 0$

$\Rightarrow -11x + 11 + 11y + 22 + 11z = 0$

$\Rightarrow x + y + z = 3$

Cartesian Form

$\vec{AR} \cdot \vec{n} = 0$

$((x-1)\hat{i} + (y-2)\hat{j} + z\hat{k})(-11\hat{i} + 11\hat{j} + 11\hat{k}) = 0$

Vector Form

points 2 remember

If 2 equation of lines are given, then to obtain the equation of a plane containing these lines can be obtained as follows:

- First find $\vec{p} \times \vec{q}$ which will provide us the normal vector of plane
- Consider the pt. A and find the equation of a plane with the help of it
- Now substitute the pt. B in equation of plane to ensure that both lines are contained by the plane

Ques: $\vec{r} = \langle 0,2,1\rangle + \lambda \langle 1,-1,1\rangle$ $\vec{r} = \langle 2,3,6\rangle + \mu \langle 2,1,5\rangle$. Find the equation of plane containing both line.

Sol: $\begin{vmatrix} (x-0) & (y-2) & (z-1) \\ 1 & -1 & 1 \\ 2 & 1 & 5 \end{vmatrix} = 0$

$\Rightarrow x(-5-1) - (y-2)(5-2) + (z-1)(1+2) = 0$

$\Rightarrow -6x - 3y + 3z + 3 = 0$

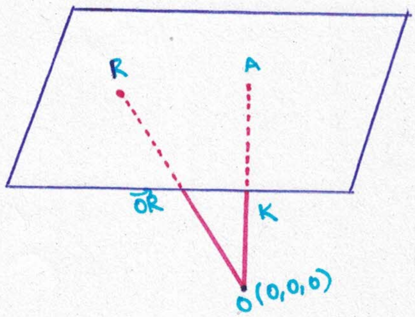
$\Rightarrow -6(2) - 3(3) + 3(6) + 3 = 0$

$\Rightarrow 0 = 0$

Plane $-$ $6x + 3y - 3z = 3 \Rightarrow 2x + y - z = 1$

EQUATION OF PLANE IN NORMAL FORM

If normal vector is given and perpendicular distance of origin from plane $= K$ is also given, then



Projection $\vec{OR}$ on $\vec{n} = K$

$\dfrac{\vec{OR} \cdot \vec{n}}{|\vec{n}|} = K$

$\vec{OR} \cdot \hat{n} = K$

$\boxed{\vec{r} \cdot \hat{n} = K}$ $K > 0$

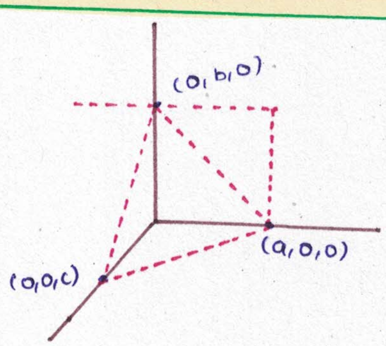
Ques: Find the DC of normal and perpendicular distance from the origin of the plane. $\vec{r}\cdot(6\hat{i}-3\hat{j}+2\hat{k}) = -1$

Sol: $\vec{r}\cdot(-6\hat{i}+3\hat{j}-2\hat{k}) = 1$

Normal unit vector $= -\frac{6}{7}\hat{i}+\frac{3}{7}\hat{j}-\frac{2}{7}\hat{k}$

Distance $= K = 1/7$

INTERCEPT FORM OF A PLANE



$$\boxed{\frac{x}{a}+\frac{y}{b}+\frac{z}{c} = 1}$$

Ques: Find the equation of a plane whose x-, y-, z- intercepts are 2, -3 and 4 respectively.

Sol: $\frac{x}{2}-\frac{y}{3}+\frac{z}{4} = 1$

POINTS TO REMEMBER

1. The equation of a plane parallel to x-axis will be given as-
$$\frac{y}{b}+\frac{z}{c} = 1$$
Similarly, plane parallel to y-axis — $\frac{x}{a}+\frac{z}{c}=1$
plane parallel to z-axis — $\frac{x}{a}+\frac{y}{b}=1$

2. Plane parallel to x-y plane- $x = c$

3. If a plane intersecting the co-ordinate axis at 3 distinct pts, then the area of the Δ formed by these 3 points will be
$$|\Delta| = \frac{1}{2}\sqrt{(ab)^2+(bc)^2+(ca)^2}$$

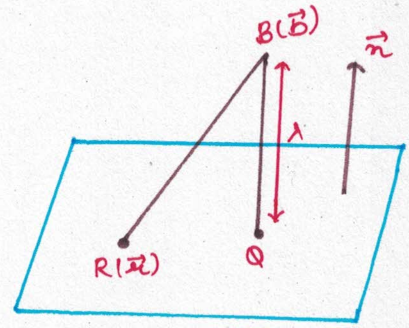
THE DISTANCE BETWEEN A POINT AND A PLANE

$|BQ|$ = Projection of $\vec{BR}$ on $\vec{n}$

$$= \left|\frac{\vec{BR}\cdot\vec{n}}{|\vec{n}|}\right|$$

$$= \left| \frac{(\vec{r}-\vec{b})\cdot\vec{n}}{|\vec{n}|} \right|$$

$$= \left| \frac{\vec{r}\cdot\vec{n} - \vec{b}\cdot\vec{n}}{|\vec{n}|} \right|$$



$$BQ = \left| \frac{\vec{b}\cdot\vec{n} - K}{|\vec{n}|} \right| \Bigg\} \text{Vector Form} \qquad (\vec{r}\cdot\vec{n} = K)$$

$\vec{r} = (x, y, z)$ $\quad \vec{n} = (a, b, c)$ $\quad \vec{b} = (x_2, y_2, z_2)$

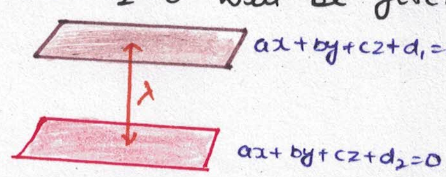
Plane - $\quad ax + by + cz + d = 0$

BQ - $\quad \dfrac{ax_2 + by_2 + cz_2 + d}{\sqrt{a^2+b^2+c^2}}$

NOTE

1. The distance between two parallel planes $P_1 - ax + by + cz + d_1 = 0$ and $P_2 - ax + by + cz + d_2 = 0$ will be given as-

$$\lambda = \left| \frac{d_1 - d_2}{\sqrt{a^2+b^2+c^2}} \right|$$



2. The distance between the planes $ax + by + cz + d = 0$ from origin will be given as- $\left| \dfrac{d}{\sqrt{a^2+b^2+c^2}} \right|$

3. The image of a pt. (x_1, y_1, z_1) with respect to plane $ax+by+cz+d = 0$ will be-

Equation of line AA' — $r' = \vec{a} + \lambda\vec{n}$

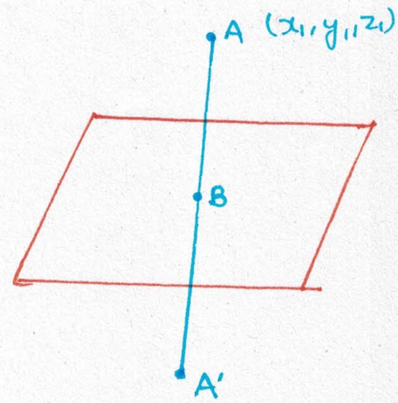
$\vec{b} = \langle x_1 + \lambda a, x_2 + \lambda b, x_3 + \lambda c \rangle$

This pt. B lies on plane, hence it will satisfy the equation of plane

$\Rightarrow$ value of λ

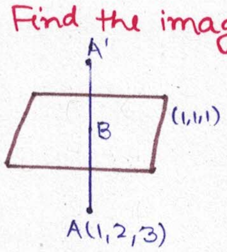
$\Rightarrow$ co-ordinate of A'

(because B is mid pt. of AA')



Ques: Find the image of the pt. (1,2,3) with respect to the plane $x+y+z+1=0$.

Sol:



$\vec{AA} = (1,2,3) + \lambda(1,1,1)$

$1 + \lambda + 2 + \lambda + 3 + \lambda = 0$

$\lambda = -7/3$

$(\vec{B}) = 1-\frac{7}{3},\ 2-\frac{7}{3},\ 3-\frac{7}{3}$

Foot of perpendicular $-\ \vec{B} = \left(-\frac{4}{3}, -\frac{1}{3}, \frac{2}{3}\right)$

$-\frac{4}{3} = \frac{1+x_1}{2}$ $\quad$ $-\frac{1}{3} = \frac{2+y_1}{2}$ $\quad$ $\frac{2}{3} = \frac{3+z}{2}$

$x_1 = -\frac{11}{3}$ $\quad$ $y_1 = -8/3$ $\quad$ $z = -5/3$

Image $-\ \left(-\frac{11}{3}, -\frac{8}{3}, -\frac{5}{3}\right)$

Ques: Find the plane parallel to $2x-6y+3z=0$ which is at a distance of 2 unit from the pt. $(1,2,-3)$

Sol: Plane $-\ 2x-6y+3z+\lambda=0$

$$\Rightarrow \frac{2(1)-6(2)+3(-3)+\lambda}{\sqrt{2^2+6^2+3^2}} = 2$$

$$\Rightarrow -19+\lambda = \pm 14$$

$$\Rightarrow \lambda = 33,\ \lambda = 5$$

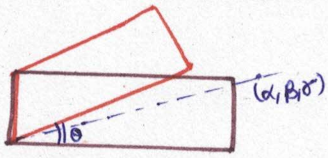
$$\left.\begin{aligned} 2x-6y+3z+33&=0 \\ 2x-6y+3z+5&=0 \end{aligned}\right\rangle \text{ solutions}$$

- The image of the pt. (x_1, y_1, z_1) with respect to the plane $ax+by+cz+d=0$ will be given as-

$$\boxed{\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c} = \frac{-2(ax_1+by_1+cz_1+d)}{a^2+b^2+c^2}}$$

- If 2 planes $a_1x+b_1y+c_1z+d_1=0$ and $a_2x+b_2y+c_2z+d_2=0$ are inter-secting planes then the angle bisector plane will be given as-

$$\boxed{\frac{a_1x+b_1y+c_1z+d_1}{\sqrt{a_1^2+b_1^2+c_1^2}} = \pm\frac{a_2x+b_2y+c_2z+d_2}{\sqrt{a_2^2+b_2^2+c_2^2}}}$$

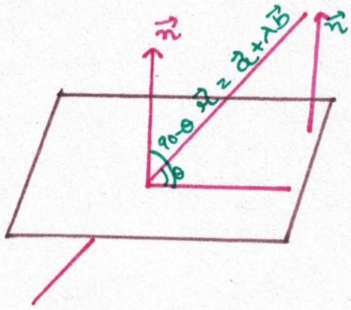


- If we wish to find the equation of plane passing through the LOI of 2 given planes, then it will be given as-

$$\boxed{P_1 + \lambda P_2 = 0}$$

Where λ is a parameter obtained by the geometrical solution provided in the question.

- The angle between a line and a plane will be calculated as follows-

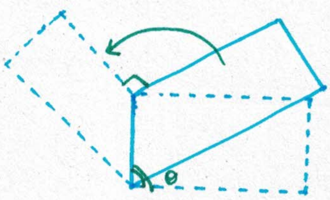


$$\vec{p}\cdot\vec{n} = |\vec{p}||\vec{n}|\cos(90-\theta)$$

$$\boxed{\sin\theta = \frac{\vec{p}\cdot\vec{n}}{|\vec{p}||\vec{n}|}}$$

Ques: $x-y-z-4=0$ is rotated through an angle about its LOI with the plane $x+y+2z-4=0$. Find the equation of that plane in new position.

Sol:



$$x-y-z-4+\lambda(x+y+z-4)=0$$

$$x(1+\lambda)+y(-1+\lambda)+z(-1+2\lambda)-4\lambda-4=0$$

$$\vec{n} = (1+\lambda)\hat{i} + (-1+\lambda)\hat{j} + (-1+2\lambda)\hat{j}$$

$$(1+\lambda)(1) - \lambda + 1 \quad -2\lambda+1=0$$

$$\lambda = 3/2$$

Equation of plane : $\frac{5}{2}\hat{i} + \frac{1}{2}\hat{j} + 4\hat{j} = 0$

Plane : $5x+y+4z-20=0$

STRAIGHT LINE

SYMMETRICAL FORM

The vector equation of line passing through (x_1, y_1, z_1) and parallel to a vector $a\hat{i}+b\hat{j}+c\hat{k}$ will be given as-

$$\vec{r} = (x_1, y_1, z_1) + \lambda(a, b, c)$$

$$(x-x_1), (y-y_1), (z-z_1) = \lambda(a, b, c)$$

Cartesian form of line-

$$\boxed{\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c} = \lambda}$$

The above equation represents symmetrical form of a line passing through (x_1, y_1, z_1) and having DR's a, b, c

POINTS TO REMEMBER

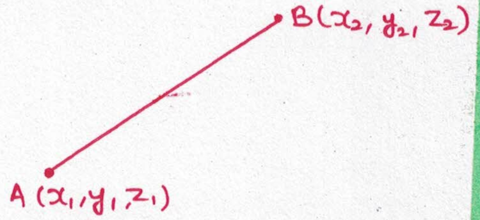
1. Any pt. on this line will have a parametric co-ordinate $(x_1+a\lambda)$, $(y_1+b\lambda)$, $(z_1+c\lambda)$

2. The symmetrical form of line passing through $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ will be given as -

$$\vec{r} = \langle x_1, y_1, z_1 \rangle + \lambda\,(x_2-x_1,\ y_2-y_1,\ z_2-z_1)$$

or

$$\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1} = \lambda$$



Ques: If the symmetrical form of line is -

$$\frac{5x-3}{7} = \frac{2y+2}{3} = \frac{1-2z}{1} = K$$

then convert it into vector form.

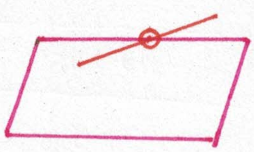
Sol:

$$\frac{x-3/5}{7/5} = \frac{y+1}{3/2} = \frac{z-1/2}{(-2)} = K$$

$$\vec{r} = \frac{3}{5} - 1 + \frac{1}{2} + K\left(\frac{7}{5} + \frac{3}{2} - \frac{1}{2}\right)$$

Ques: Find the distance of the pt. $(-1, -5, -10)$ from the pt. of intersection of the line $\frac{x-2}{2} = \frac{y+1}{4} = \frac{z-2}{12}$ with plane $x-y+z=5$

Sol: $2x+4y+12z+d=0$



$x = 2+2\lambda$

$y = -1+4\lambda$

$z = 2+12\lambda$

$2+2\lambda+1-4\lambda+2+12 = 5$

$\lambda = 0$

$x = 2, -1, 2$

$(3)^2+(4)^2+(12)^2$

Distance $= \sqrt{169} = 13$

UNSYMMETRICAL FORM

As we know if the 2 planes intersect, then they will provide a LOI which is known as unsymmetrical form of line, given as-

$$\boxed{L:\ a_1x+b_1y+c_1z+d_1 = 0 = a_2x+b_2y+c_2z+d_2}$$

The LOI will cut x-y plane or y-z plane or z-x plane. So substitute 1 co-ordinate zero in the both the equation of plane to obtain the pt. from where the line is passing.

Now, we have pt. of passing and DR's of parallel vector by $(\vec{n_1} \times \vec{n_2})$. So we can convert it into symmetrical form of line.

Ques: $x + y - 2z - 8 = 0 = 3x - y + 4z + 12$

Sol: $\vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} 1 & 1 & -2 \\ 3 & -1 & 4 \end{vmatrix}$

$= \hat{i}(2) - \hat{j}(10) + \hat{k}(-4)$

$= 2\hat{i} - 10\hat{j} - 4\hat{k}$

Put $z=0$ in both planes

$x+y=8 \qquad 3x-y=12$

$x=5 \ , \ y=3$

Symmetrical: $\dfrac{x-5}{2} = \dfrac{y-3}{-10} = \dfrac{z-0}{-4} = K$

Ques: Find the line passing through $(1,4,-2)$ and parallel to the planes $6x+2y+2z+3=0$ and $x+2y-6z+4=0$

Sol:

$\vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} 6 & 2 & 2 \\ 1 & 2 & -6 \end{vmatrix}$

$= \hat{i}(-12-4) - \hat{j}(-36-2) + \hat{k}(12-2)$

$= -16\hat{i} + 38\hat{j} + 10\hat{k}$

$\dfrac{x-1}{-16} = \dfrac{y-4}{38} = \dfrac{z+2}{10} = K$

Ques: Find the line passing through $(2,-1,-1)$ and parallel to the planes $4x+y+z+2=0$ and perpendicular to the line $2x+y=0=x-y+z$

Sol. $\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & 0 \\ 1 & -1 & 1 \end{vmatrix} = i - \hat{j}(2) + \hat{k}(-3)$

$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & 1 & 1 \\ 1 & -2 & -3 \end{vmatrix} = \hat{i}(-3+2) - \hat{j}(-12-1) + \hat{k}(-8-1)$

$= -\hat{i} - 13\hat{j} - 9\hat{k}$

$\dfrac{x-2}{-1} = \dfrac{y+1}{-13} = \dfrac{z+1}{-9}$

Ques: Find the distance of a pt. $(1,0,-3)$ from the plane $x-y-z=9$, measured parallel to the line $\dfrac{x-2}{2} = \dfrac{y+2}{3} = \dfrac{z+3}{-6} = K$ as well as find the distance of same pt. from the given line along the given plane.

Sol: $\vec{n}$ of $2\hat{i}+3\hat{j}-6\hat{k}$ $\qquad$ $\vec{n}$ of plane $= \hat{i}+\hat{j}-\hat{k}$

AB plane $\rightarrow$ $2x+3y-6z+d=0$

$= \dfrac{2-18+65}{\sqrt{49}} = \dfrac{49}{7} = 7$

$L_{AB} = \vec{r} = \langle 1,0,3\rangle + \lambda\langle 2,3,-6\rangle$

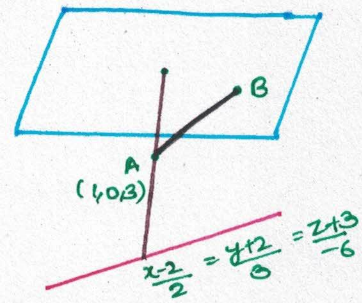
PV of B $= (1+2\lambda,\ 3\lambda,\ -3-6\lambda)$ lies on the plane

$1+2\lambda-3\lambda+3+6\lambda = 9$

$5\lambda = 5 \Rightarrow \lambda = 1$

pt. $(3,3,-9)$

Distance $= \sqrt{4+9+36} = 7$



$D = (2+2k,\ -2+3k,\ -3-6k)$

$\vec{AD} = \langle 1+2k,\ -2+3k,\ -6k\rangle$

$\vec{AD}\cdot\vec{n} = 0$

$k = -3/5$

Ques: Find the equation of a plane containing the line $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ and parallel to the line $\frac{x-2}{3} = \frac{y-4}{4} = \frac{z-5}{5}$

Sol: (I) $\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 4 & 5 \\ 2 & 3 & 4 \end{vmatrix} = \hat{i}(16-15) - \hat{j}(12-10) + \hat{k}(9-8)$

$= \hat{i} - 2\hat{j} + \hat{k}$

Plane $\rightarrow$ $x-2y+z+d=0$

$1-4+3+d=0$

$d=0$

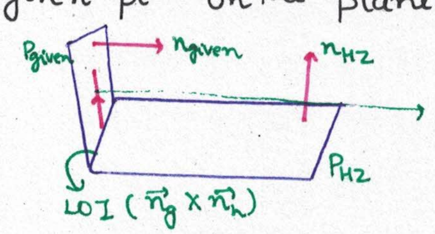
$x-2y+z=0$

(II) $(x-1,\ y-2,\ z-3)$, $(2,3,4)$, $(3,4,5)$ are coplanar

$\begin{vmatrix} x-1 & y-2 & z-3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix} = 0$

The Line of Greatest Slope of a given plane

It is a line in a given plane and perpendicular to the line of intersection of the given plane with the horizontal plane and passing through a given pt. on the plane.



Line will be parallel to $\vec{n_g} \times (\vec{n_g} \times \vec{n_h})$

pass it through the given pt.

Ques: Find the equation of plane passing through the pts. (3, 4, 1) & (0, 1, 0) & parallel to the line $\frac{x+3}{2} = \frac{y-3}{7} = \frac{z-2}{5}$

Sol: Line$_1$ = $\frac{x-3}{-3} = \frac{y-4}{-3} = \frac{z-1}{-1}$

Line$_2$ = $\frac{x+3}{2} = \frac{y-3}{7} = \frac{z-2}{5}$

$$\begin{vmatrix} (x-3) & (y-4) & (z-1) \\ -3 & -3 & -1 \\ 2 & 7 & 5 \end{vmatrix} = 0$$

$\Rightarrow (x-3)(-15+7) - (y-4)(-15+2) + (z-1)(-21+6) = 0$

$\Rightarrow -8x + 13y - 15z + 24 - 52 + 15 = 0$

$\Rightarrow -8x + 13y - 15z - 18 = 0$